1. Let $A = \begin{pmatrix} 3 & 4 & -2 & 0 \\ 6 & 8 & -2 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & -1 & -3 \end{pmatrix}$,

   (a) Perform elimination on A to obtain an upper triangular matrix U.

   (b) Find a permutation matrix P such that there is a matrix L such that $PA = LU$. (You do not need to find L).

   (c) What is the determinant of U?

   (d) Is A invertible? Explain how you know.

   (e) What is the rank of A and the dimension of each of the four fundamental subspaces?

   (f) Given a vector $\mathbf{b} \in \mathbb{R}^4$, how many solutions could the equation $A\mathbf{x} = \mathbf{b}$ have? Explain why in each case.

   Solution:

   (a) Beginning with the matrix A, there is a pivot (3) in Row 1 Column 1, so we subtract twice Row 1 from Row 2 to obtain

   $$\begin{pmatrix} 3 & 4 & -2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & -1 & -3 \end{pmatrix}.$$

   Now there is no pivot in Row 2 Column 2, so we swap rows 2 and 3 to obtain

   $$\begin{pmatrix} 3 & 4 & -2 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & -1 & -3 \end{pmatrix}.$$

   Now there is a pivot (-1) in Row 2 Column 2, so we add Row 2 to Row 4 to obtain

   $$\begin{pmatrix} 3 & 4 & -2 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

   This is an upper triangular matrix U.

   (b) Since in Part (a) we swapped Rows 2 and 3 in the second step, P is the 4×4 permutation matrix that switches these rows:

   $$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the record, we obtain the following $PA = LU$ decomposition:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 & -2 & 0 \\ 6 & 8 & -2 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 & -2 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(c) Since the determinant of an upper triangular matrix is the product of the entries on the diagonal, the determinant of U is $(3)(-1)(2)(0) = 0$.

(d) Since the determinant of A is equal to $\det P \det L \det U = 0$, A is not invertible.

(e) The rank of A is 3.

subspace	dimension
$C(A)$	$r = 3$
$C(A^T)$	$r = 3$
$N(A)$	$n - r = 1$
$N(A^T)$	$m - r = 1$

(f) Since A does not have full row rank, $A\mathbf{x} = \mathbf{b}$ may have no solutions.
If $A\mathbf{x} = \mathbf{b}$ does have a solution, since the nullspace is 1-dimensional, there will be an infinite number of solutions.

2. Let $E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and $E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and suppose $A = E_1E_2E_3E_4$.

(a) Write an expression for the inverse of A (you may write this as a product of matrices, no need to simplify), or explain why A is not invertible.

(b) What is the determinant of A?

(c) Does the transformation A preserve or reverse orientation, or neither?

(d) Suppose a region with 4-dimensional volume 10 is transformed by A. What is the volume of the resulting region?

(e) Given a vector $\mathbf{b} \in \mathbb{R}^4$, how many solutions could the equation $A\mathbf{x} = \mathbf{b}$ have? Explain why in each case.

Solution:

(a) Since $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}$, $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

and $E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

$$A^{-1} = E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}.$$

(b) Since each matrix E_i is triangular, its determinant is the product of the entries on the diagonal, so $\det E_1 = 1$, $\det E_2 = 4$, $\det E_3 = 1$, and $\det E_4 = 1$. Since the determinant is multiplicative,

$$\det A = \det E_1 \det E_2 \det E_3 \det E_4 = 1 \cdot 4 \cdot 1 \cdot 1 = 4.$$

(c) Since $\det A > 0$, A preserves orientation.

(d) The volume of the resulting region is $4 \cdot 10 = 40$.

(e) Since $\det A \neq 0$, A is invertible. Thus for every $\mathbf{b} \in \mathbb{R}^4$, the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

3. Let $\mathbf{v}_1 = (1, 1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0, 0)$, and $\mathbf{v}_3 = (0, 1, 1, 0)$, and let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

(a) Use the Gram-Schmidt algorithm to produce an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ for V.

(b) What is the dimension of V?

(c) Which of the fundamental subspaces of A is V?

(d) Find the QR decomposition for the matrix A.

(e) Find the projection of $\mathbf{v}_3$ onto $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Solution:

(a) First we find an orthogonal basis $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$ for V:
First vector: Let $\mathbf{A}_1 = \mathbf{v}_1 = (1, 1, 1, 1)$.

Second vector: Let

$$\begin{aligned}\mathbf{A}_2 &= \mathbf{v}_2 - \text{proj}_{\mathbf{A}_1}\mathbf{v}_2 \\ &= \mathbf{v}_2 - \frac{\mathbf{A}_1^T\mathbf{v}_2}{\mathbf{A}_1^T\mathbf{A}_1}\mathbf{A}_1 \\ &= \begin{pmatrix}1\\1\\0\\0\end{pmatrix} - \frac{2}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \\ &= \begin{pmatrix}1/2\\1/2\\-1/2\\-1/2\end{pmatrix}\end{aligned}$$

Third vector: Let

$$\begin{aligned}\mathbf{A}_3 &= \mathbf{v}_3 - \text{proj}_{\mathbf{A}_1}\mathbf{v}_3 - \text{proj}_{\mathbf{A}_3}\mathbf{v}_3 \\ &= \mathbf{v}_3 - \frac{\mathbf{A}_1^T\mathbf{v}_3}{\mathbf{A}_1^T\mathbf{A}_1}\mathbf{A}_1 - \frac{\mathbf{A}_2^T\mathbf{v}_3}{\mathbf{A}_2^T\mathbf{A}_2}\mathbf{A}_2 \\ &= \begin{pmatrix}0\\1\\1\\0\end{pmatrix} - \frac{2}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{0}{1}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} \\ &= \begin{pmatrix}-1/2\\1/2\\1/2\\-1/2\end{pmatrix}\end{aligned}$$

Now that we have an orthogonal basis $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$, for $1 \leq i \leq 3$ we let $\mathbf{q}_i = \mathbf{A}_i/\|\mathbf{A}_i\|$ to obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$. Note $\|\mathbf{A}_1\| = 2$, while $\|\mathbf{A}_1\| = \|\mathbf{A}_2\| = 1$.

$$\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\} = \{\mathbf{A}_1/2, \mathbf{A}_2/1, \mathbf{A}_3/1\} = \left\{\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}, \begin{pmatrix}1/2\\1/2\\-1/2\\-1/2\end{pmatrix}, \begin{pmatrix}-1/2\\1/2\\1/2\\-1/2\end{pmatrix}\right\}$$

(b) The subspace V is 3-dimensional.

(c) The subspace V is the column space of A.

(d)

$$A = \begin{pmatrix}1&1&0\\1&1&1\\1&0&1\\1&0&0\end{pmatrix} = \begin{pmatrix}\mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3\end{pmatrix}\begin{pmatrix}\mathbf{q}_1^T\mathbf{v}_1 & \mathbf{q}_1^T\mathbf{v}_2 & \mathbf{q}_1^T\mathbf{v}_3\\0 & \mathbf{q}_2^T\mathbf{v}_2 & \mathbf{q}_2^T\mathbf{v}_3\\0&0&\mathbf{q}_3^T\mathbf{v}_3\end{pmatrix} = \begin{pmatrix}1/2&1/2&-1/2\\1/2&1/2&1/2\\1/2&-1/2&1/2\\1/2&-1/2&-1/2\end{pmatrix}\begin{pmatrix}2&1&1\\0&1&0\\0&0&1\end{pmatrix}$$

(e) Since $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{q}_1, \mathbf{q}_2\}$, and $\mathbf{q}_1$ and $\mathbf{q}_2$ are orthogonal, the projection of $\mathbf{v}_3$ onto $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is the sum

$$\text{proj}_{\mathbf{q}_1}\mathbf{v}_3 + \text{proj}_{\mathbf{q}_2}\mathbf{v}_3 = \frac{\mathbf{q}_1^T\mathbf{v}_3}{\mathbf{q}_1^T\mathbf{q}_1}\mathbf{q}_1 + \frac{\mathbf{q}_2^T\mathbf{v}_3}{\mathbf{q}_2^T\mathbf{q}_2}\mathbf{q}_2 = \frac{1}{1}\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix} + \frac{0}{1}\begin{pmatrix}1/2\\1/2\\-1/2\\-1/2\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}.$$

4. Suppose M is an $n \times n$ Markov matrix, and let $\mathbf{1} = (1, \ldots, 1)$ be the vector with n components, each of which is 1. Label each of the following statements true or false, with a brief reason for each.

   (a) M has linearly independent columns.

   (b) One of the eigenvectors for M is $\mathbf{1}$.

   (c) One of the eigenvectors for M^T is $\mathbf{1}$.

   (d) If $\mathbf{x}$ is a unit vector, then $M\mathbf{x}$ is a unit vector.

   (e) If the sum of the components of $\mathbf{x}$ is 7, then the sum of the components of $M^3\mathbf{x}$ is 7.

   (f) If P is an $n \times n$ permutation matrix, then MP is a Markov matrix.

   (g) If P is an $n \times n$ permutation matrix, then PM is a Markov matrix.

   (h) The determinant of M is at most 1.

   Solution:

   (a) M has linearly independent columns.
   FALSE: $\begin{pmatrix}1/2 & 1/2\\1/2 & 1/2\end{pmatrix}$ is a Markov matrix with dependent columns.

   (b) One of the eigenvectors for M is $\mathbf{1}$.
   FALSE. If $M = \begin{pmatrix}1/3 & 1/4\\2/3 & 3/4\end{pmatrix}$, then $M\mathbf{1} = (7/12, 17/12)$, which is not a scalar multiple of $\mathbf{1}$, so $\mathbf{1}$ is not an eigenvector.

   (c) One of the eigenvectors for M^T is $\mathbf{1}$.
   TRUE. Since every row of M^T sums to 1, $M^T\mathbf{1} = \mathbf{1}$, so $\mathbf{1}$ is an eigenvector with eigenvalue 1.

   (d) If $\mathbf{x}$ is a unit vector, then $M\mathbf{x}$ is a unit vector.
   FALSE. Consider the Markov matrix $M = \begin{pmatrix}1/2 & 1/2\\1/2 & 1/2\end{pmatrix}$ and the unit vector $\mathbf{x} = (1, 0)$. Then $M\mathbf{x} = (1/2, 1/2)$, which is not a unit vector.

   (e) If the sum of the components of $\mathbf{x}$ is 7, then the sum of the components of $M^3\mathbf{x}$ is 7.
   TRUE. IF M is a Markov matrix, then M^3 is also a Markov matrix. Since multiplication by a MArkov matrix preserves component sums, the statement is true.

(f) If P is an $n \times n$ permutation matrix, then MP is a Markov matrix.
TRUE. MP is the same matrix as M, but with permuted columns. Thus all entries are still nonnegative and the sum of the entries in each column is still 1.

(g) If P is an $n \times n$ permutation matrix, then PM is a Markov matrix.
TRUE. PM is the same matrix as M, but with permuted rows. Thus all entries are still nonnegative and the sum of the entries in each column is still 1.

(h) The determinant of M is at most 1.
TRUE. The determinant of M is equal to the product of its eigenvalues. Since the eigenvalues have magnitude at most 1, their product is at most 1.

5. Suppose S is a symmetric matrix with eigenvectors $\mathbf{v}_1 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\mathbf{v}_2 = \left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$, with eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 0$.

(a) Find a diagonalization for S.

(b) Find a matrix A such that $S = A^T A$ or explain why it's not possible.

(c) Find eigenvectors, eigenvalues, and a diagonalization for S^{10}.

(d) Suppose $\mathbf{w} = -3\mathbf{v}_1 + 2\mathbf{v}_2$. Find $S^{10}\mathbf{w}$.

(e) Find a basis for each of the four fundamental subspaces for S.

Solution:

(a) Since S is symmetric, by the Spectral theorem it has a diagonalization of the form $S = Q\Lambda Q^T$:

$$S = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix}$$

(b) It is possible, because S is positive semi-definite.
Since $\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$,

$$\begin{aligned} S &= \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix} \\ &= \left[\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix} \right]^T \left[\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix} \right] \end{aligned}$$

Thus if

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix},$$

Then $S = A^T A$.

(c) The matrix S^{10} has eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, with eigenvalues 4^{10} and $0^{10} = 0$. It has diagonalization

$$S^{10} = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 4^{10} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix}$$

(d)

$$\begin{aligned} S^{10}\mathbf{w} &= S^{10}(-3\mathbf{v}_1 + 2\mathbf{v}_2) \\ &= -3S^{10}\mathbf{v}_1 + 2S^{10}\mathbf{v}_2 \\ &= -3 \cdot 4^{10}\mathbf{v}_1 + 2 \cdot 0\mathbf{v}_2 \\ &= -3 \cdot 4^{10} \left(\frac{\sqrt{3}}{2}, \frac{1}{2} \right) \end{aligned}$$

(e) Since S is symmetric, $C(S) = C(S^T)$, and $N(S) = N(S^T)$. Since S has one eigenvector with nonnegative eigenvalue, and one with eigenvalue 0, each of the fundamental subspaces are 1-dimensional.

A basis for the column space and row space is $\{\mathbf{v}_1\} = \left\{ \left(\frac{\sqrt{3}}{2}, \frac{1}{2} \right) \right\}$, and a basis for the null space (and left null space) is $\{\mathbf{v}_2\} = \left\{ \left(\frac{1}{2}, \frac{-\sqrt{3}}{2} \right) \right\}$.

6. Suppose A is the matrix with singular value decomposition

$$A = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}$$

(a) How many rows and columns does A have?

(b) What is the rank r of A?

(c) Express A as the sum of r rank 1 matrices. [You can write expressions for these matrices in terms of the singular values and singular vectors. You do not need to simplify.]

(d) Find the dimension for each of the four fundamental subspaces for A, and a basis for each one.

(e) Given a vector $\mathbf{b} \in \mathbb{R}^4$, how many solutions could the equation $A\mathbf{x} = \mathbf{b}$ have? Explain why in each case.

(f) Write a diagonalization for A^TA.

(g) Write a diagonalization for AA^T.

Solution:

(a) The matrix A has 4 rows and 2 columns.

(b) The matrix A has rank $r = 2$.

(c)

$$A = 6\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 3/5 & 4/5 \end{pmatrix} + 3\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} -4/5 & 3/5 \end{pmatrix}$$

(d) Since A is 4×2 with rank $r = 2$ ($m = 4, n = 2$),

subspace	dimension	basis
$C(A)$	$r = 2$	$\{(1/\sqrt{2}, 1/\sqrt{2}, 0, 0), (-1/\sqrt{2}, 1/\sqrt{2}, 0, 0)\}$
$C(A^T)$	$r = 2$	$\{(3/5, 4/5), (-4/5, 3/5)\}$
$N(A)$	$n - r = 0$	$\emptyset$
$N(A^T)$	$m - r = 2$	$\{(0, 0, 1/\sqrt{2}, -1/\sqrt{2}), (0, 0, 1/\sqrt{2}, 1/\sqrt{2})\}$

(e) Since A does not have full row rank, the the equation $A\mathbf{x} = \mathbf{b}$ could have no solutions. Since A has full column rank ($N(A) = \{\mathbf{0}\}$), if $A\mathbf{x} = \mathbf{b}$ has a solution, there is only one.

(f)

$$A^TA = (V\Sigma^TU^T)(U\Sigma V^T) = V(\Sigma^T\Sigma)V^T = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}\begin{pmatrix} 36 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}$$

(g)

$$\begin{aligned} AA^T &= (U\Sigma V^T)(V\Sigma^TU^T) \\ &= U(\Sigma\Sigma^T)U^T \\ &= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 36 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \end{aligned}$$